

PHY 456 Lecture 10:

Time Dependent Perturbation Theory

Time Dependent Perturbation Theory

Set up: $\hat{H} = \hat{H}_0 + \lambda \hat{H}_1(t)$

Time-dependent spectrum,
Known states of $\hat{H}_0$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = (\hat{H}_0 + \lambda \hat{H}_1(t)) |\psi\rangle$$

in SE, but energy is no longer conserved.

Let the $t=0$ state be some $|\psi_i\rangle$ (could be eigenstate of $\hat{H}_0$),

but what's the probability that $|\psi_i\rangle$ is found in another state

$|\psi_f\rangle$? Assume that both $|\psi_i\rangle, |\psi_f\rangle$ are eigenstates of $\hat{H}_0$.

Ans. $P(|\psi_i\rangle \text{ at } t=0, |\psi_f\rangle \text{ at } t) = |\langle \psi_f | \tilde{\psi}_i \rangle|^2$

where we assume ψ_i, ψ_f are 1-normalized and

$\tilde{\psi}_i(t)$ denotes the solution of

$$i\hbar \frac{\partial}{\partial t} |\tilde{\psi}_i\rangle = (\hat{H}_0 + \lambda \hat{H}_1) |\tilde{\psi}_i\rangle$$

such that $|\tilde{\psi}_i(t=0)\rangle = |\psi_i\rangle$.

Again, we said $\hat{H}_0 |\psi_i\rangle = E_i |\psi_i\rangle$

$$\hat{H}_0 |\psi_f\rangle = E_f |\psi_f\rangle$$

and stationary states $\hat{H}_0 |\psi_n\rangle = E_n |\psi_n\rangle \rightarrow \{|\psi_n\rangle, \psi_n\}$ form a complete set.

Since $|\psi_n\rangle$ are complete eigenstates basis, we can expand any state in terms of stationary solutions:

$$|\tilde{\psi}_i(t)\rangle = \sum_n C_n(t) |\psi_n\rangle \quad (1)$$

with boundary condition $C_n(0) = \delta_{ni}$.

The Schrödinger equation implies

$$\sum_n i\hbar \dot{C}_n |\psi_n\rangle = \sum_n (C_n(t) E_n |\psi_n\rangle + C_n(t) \lambda \hat{H}_1(t) |\psi_n\rangle)$$

$$\Rightarrow \sum_n (i\hbar \dot{C}_n(t) - E_n C_n(t)) |\psi_n\rangle = \lambda \sum_n C_n(t) \hat{H}_1(t) |\psi_n\rangle \quad (2)$$

Without loss of generality, let $C_n(t) = e^{-i\frac{E_n t}{\hbar}} d_n(t)$. Then

$$i\hbar \dot{C}_n - E_n C_n = e^{-i\frac{E_n t}{\hbar}} \dot{d}_n(t)$$

which implies

$$\sum_n i\hbar e^{-iE_n t/\hbar} \dot{d}_n(t) |\psi_n\rangle = \lambda \sum_n e^{-iE_n t/\hbar} d_n(t) \hat{H}_1(t) |\psi_n\rangle \quad (3)$$

Taking $\langle \psi_m |$ inner product under completeness,

$$\sum_n i\hbar e^{-iE_n t/\hbar} \dot{d}_n(t) \delta_{nm} = \lambda \sum_n e^{-iE_n t/\hbar} d_n(t) \langle \psi_m | \hat{H}_1 | \psi_n \rangle$$

$$\Rightarrow \dot{d}_m(t) = -\frac{i}{\hbar} \sum_n e^{-i\frac{(E_n - E_m)t}{\hbar}} \langle \psi_m | \lambda \hat{H}_1(t) | \psi_n \rangle d_n(t)$$

This looks similar to our PT before, so we'll expand $d_n(t)$ in a series in λ :

$$d_k(t) = d_k^0(t) + \lambda d_k^1(t) + \lambda^2 d_k^2(t) + \dots$$

$$\text{At } \mathcal{O}(\lambda^0), \quad \dot{d}_k(t) = 0 \quad \text{or} \quad d_k^0 = \text{constant},$$

and initial condition has $d_k^0 = \delta_{ni}$.

$$\mathcal{O}(\lambda^1), \quad \dot{d}_m^1(t) = -\frac{i}{\hbar} \sum_n e^{-i\frac{(E_n - E_m)t}{\hbar}} \langle \psi_m | \hat{H}_1(t) | \psi_n \rangle \frac{d_n^0(t)}{\delta_{ni}}$$

$$\Rightarrow \dot{d}_m^1(t) = -\frac{i}{\hbar} e^{i\omega_{mi}t} \langle \psi_m | \hat{H}_1(t) | \psi_i \rangle$$

$$\text{where } \omega_{mi} = \frac{E_m - E_i}{\hbar}.$$

Integrating with $d_m^1(0) = 0$ yields

$$d_m^1(t) = -\frac{i}{\hbar} \int_0^t dt' e^{-i\omega_{mi}t'} \langle \psi_m | \hat{H}_1(t') | \psi_i \rangle$$

The first order correction, with $c_n(t) = e^{-i E_n t / \hbar} d_n(t)$, implies that

$$|\tilde{\psi}_i(t)\rangle = e^{-i \frac{E_i}{\hbar} t} |\psi_i\rangle - \frac{i}{\hbar} \lambda \sum_n e^{-i \frac{E_n}{\hbar} t} \int_0^t dt' e^{i \omega_{ni} t'} |\psi_n\rangle \langle \psi_n | \hat{H}_1(t') | \psi_i \rangle$$

← equation (3)

The probability of transition between $|\psi_i\rangle \rightarrow |\psi_f\rangle$ is then

$$|\langle \psi_f | \psi_i \rangle|^2 = \left| \delta_{fi} e^{-i E_i t / \hbar} - \frac{i}{\hbar} \lambda e^{-i E_f t / \hbar} \int_0^t e^{i \omega_{fi} t'} dt' \langle \psi_f | \hat{H}_1(t') | \psi_i \rangle \right|^2$$

$$P_{i \rightarrow f}(t) \stackrel{f \neq i}{=} \frac{\lambda^2}{\hbar^2} \left| \int_0^t e^{i \omega_{fi} t'} dt' \langle \psi_f | \hat{H}_1(t') | \psi_i \rangle \right|^2$$

Now, what if we turn on the interaction only for a finite amount of time?



We're really interested in the $t \rightarrow \infty$ behaviour ($t > T$):

$$P_{i \rightarrow f}(t > T) = \frac{\lambda^2}{\hbar^2} \left| \int_0^\infty e^{i\omega_f t} \langle \psi_f | \hat{H}_1(t) | \psi_i \rangle dt \right|^2$$

↪ ω_f Fourier transform component of $\langle \psi_f | \hat{H}_1(t) | \psi_i \rangle$ matrix element.

Questions: - When is keeping $\mathcal{O}(\lambda)$ ok?

- what can we do with this?

SHO Perturbation

$$\text{Let } \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \quad \text{with} \quad \hat{H}_1(t) = -q \mathcal{E}_0 \hat{x} e^{-t^2/\tau^2}$$

↗
electric field which only works on timescale $\sim \tau$, $E(t) \rightarrow 0$ as $t \rightarrow \pm\infty$.

We will treat $\hat{H}_1(t)$ perturbatively and find

2) It's effect, assuming at $t_0 \rightarrow -\infty$ the

system is in ground state $|0\rangle$

b) When the perturbation is ok...

Let's start with our derivation

$$P_{i \rightarrow f}(t_0, t) = \frac{\lambda^2}{\hbar^2} \left| \int_{t_0}^t dt' e^{i\omega_{fi}t'} \langle \psi_f | \hat{H}_1(t') | \psi_i \rangle \right|^2$$

and $|\psi_i\rangle = |0\rangle$. Since $\lambda \hat{H}_1 = -q \mathcal{E}_0 e^{-t^2/\tau^2} \hat{x}$

and $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^\dagger + \hat{a})$, $\langle \psi_f |$ can only be $\langle 1 |$
from $\hat{a}^\dagger |0\rangle = |1\rangle$.

$$\text{Thus } \langle 1 | \hat{x} | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

We find that

$$P_{|0\rangle \rightarrow |1\rangle}(-\infty, t) = \frac{q^2 \mathcal{E}_0^2}{\hbar^2} \frac{\hbar}{2m\omega} \left| \int_{-\infty}^t e^{-i\omega_{fi}t'} e^{-t'^2/\tau^2} dt' \right|^2$$

↑
doable (get an erf(t)).

But let's instead ask of the total probability:

$$P_{|0\rangle \rightarrow |1\rangle}(-\infty, \infty) = \frac{q^2 \mathcal{E}_0^2}{2m\omega \hbar} \left| \int_{-\infty}^{\infty} dt e^{-i\omega_{fi}t - t^2/\tau^2} \right|^2$$

↑
can integrate by completing the square, we find

$$= \tau \sqrt{\pi} e^{-\frac{\omega_{fi}^2 \tau^2}{4}}$$

$$= \frac{q^2 \epsilon_0^2 \pi}{2m\omega t_0} \pi^2 e^{-\omega_{fi}^2 \pi^2 / 2}$$

which is equivalently the $P_{10 \rightarrow 11}$ ($t - t_0 \gg \tau$) limit.

→ For the answer to be sensible, however,

$$\frac{q^2 \epsilon_0^2 \pi}{2m\omega t_0} \pi^2 e^{-\omega_{fi}^2 \pi^2 / 2} \ll 1,$$

So how can we interpret it?

$q\epsilon_0 \tilde{x} \sim$ max energy of perturbation

with $\tilde{x} \sim \sqrt{\frac{\hbar}{2m\omega}} \sim$ width of ground state

$$P_{10 \rightarrow 11} \sim \pi \cdot \frac{q^2 \epsilon_0^2 t_0}{2m\omega} \cdot \frac{1}{(t_0\omega)^2} \cdot \omega^2 \pi^2 \cdot e^{-\omega^2 \pi^2 / 2}$$

$$\sim \left[\frac{\text{Max energy of perturbation}}{\text{ground state energy}} \right] \cdot \frac{\pi^2}{T^2} e^{-\left(\frac{\pi^2}{T^2}\right) \cdot \frac{1}{2}}$$



Max strength of
E-field over
unperturbed energy

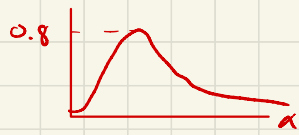


T is oscillator period
(natural timescale
of system)

Implying that ϵ_0 and τ can be varied independently.

But in this problem, observe that

$$\alpha^2 e^{-\alpha^2/2} < 1 \quad \forall \alpha$$



So the condition is really that $\frac{q E_0 \sqrt{\hbar}}{\sqrt{m \omega}} \frac{1}{\hbar \omega} \ll 1$
 or that $q E_0$ is "small"

Periodic Perturbation

$$P_{i \rightarrow f}(t_0, t) = \frac{\lambda^2}{\hbar^2} \left| \int_{t_0}^t dt' e^{i\omega_{fi}t'} \langle \psi_f | \hat{H}_1(t') | \psi_i \rangle \right|^2$$

is the "master derivation". Let $\hat{H}_1(t) = \hat{A} e^{i\omega t} + \hat{A}^\dagger e^{-i\omega t}$

and let $t_0 = 0$:

$\hat{A}, \hat{A}^\dagger$ time-independent
 operators spanned
 by eigenvectors of $\hat{H}_0$.

$$\begin{aligned} P_{i \rightarrow f}(t) &= \frac{\lambda^2}{\hbar^2} \left| \int_0^t dt' e^{i\omega_{fi}t'} \left[e^{i\omega t'} \langle \psi_f | \hat{A} | \psi_i \rangle + e^{-i\omega t'} \langle \psi_f | \hat{A}^\dagger | \psi_i \rangle \right] \right|^2 \\ &= \frac{\lambda^2}{\hbar^2} \left[A_{fi} \frac{1}{i(\omega_{fi} + \omega)} e^{i(\omega_{fi} + \omega)t'} \right]_0^t + A_{if}^* \frac{1}{i(\omega_{fi} - \omega)} e^{i(\omega_{fi} - \omega)t'} \Big|_0^t \Big|^2 \end{aligned}$$

with $A_{fi} \equiv \langle \psi_f | \hat{A} | \psi_i \rangle$, $A_{fi}^\dagger = \langle \psi_f | \hat{A}^\dagger | \psi_i \rangle = (A_{if}^*)$

$$\begin{aligned}
&= \frac{\lambda^2}{\hbar^2} \left[\frac{-i A_{fi}}{\omega_f + \omega} \left(e^{i(\omega_f + \omega)t} - 1 \right) - \frac{i A_{if}^*}{\omega_f - \omega} \left(e^{i(\omega_f - \omega)t} - 1 \right) \right]^2 \\
&= \frac{\lambda^2}{\hbar^2} \left[A_{fi} \frac{2}{\omega_f + \omega} \sin \frac{(\omega_f + \omega)t}{2} e^{i(\omega_f + \omega)t/2} + A_{if}^* \frac{2}{\omega_f - \omega} \sin \frac{(\omega_f - \omega)t}{2} e^{i(\omega_f - \omega)t/2} \right]^2 \\
&= \frac{4\lambda^2}{\hbar^2} \left[A_{fi} e^{i\omega_+ t/2} \frac{\sin(\omega_+ t/2)}{\omega_+} + A_{if}^* e^{i\omega_- t/2} \frac{\sin(\omega_- t/2)}{\omega_-} \right]^2
\end{aligned}$$

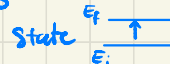
where $\omega_{\pm} = \omega_f \pm \omega$ (Very general $\Rightarrow$ hard to unpack)

Let's now examine some cases:

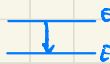
1) Resonances:

• $\omega > 0$, and

$$\omega \simeq \omega_f = E_f - E_i$$

transition to $E_f > E_i$ state 

$$\omega \simeq -\omega_f = E_i - E_f$$

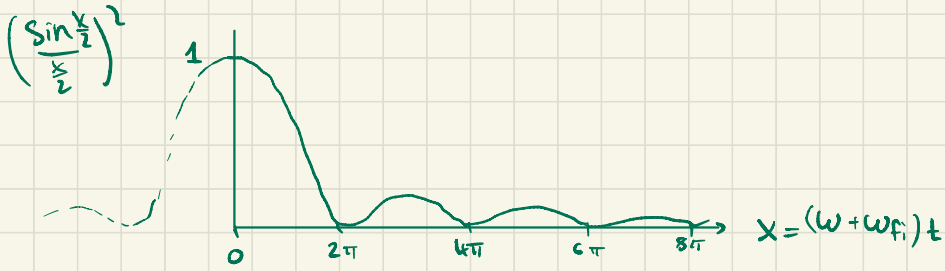
transition to $E_f < E_i$ state 

Let's examine $\omega_+ \simeq 0$ (lower E transition) and $\omega_- = \omega_+ - 2\omega_f$

so $\omega \simeq -\omega_f$:

$$P_{i \rightarrow f}(t) = \frac{4\lambda^2}{\hbar^2} |A_{fi}|^2 \left[\frac{\sin \frac{(\omega_+ - \omega_f)t}{2}}{(\omega_+ - \omega_f)} \right]^2$$

$$= \frac{\lambda^2}{\hbar^2} t^2 |A_{fi}|^2 \left[\frac{\sin \frac{(\omega_f - \omega_i) t}{2}}{(\omega_f - \omega_i) t / 2} \right]^2$$



$$-\frac{2\pi}{t} \leq \omega_f - \omega_i \leq \frac{2\pi}{t}$$

We call $\omega_f - \omega_i = \Delta\omega = \frac{4\pi}{t}$ the "width" of the resonance

When $\Delta\omega \approx 0$, $P_{i \rightarrow f}(t) \approx \frac{\lambda^2 t^2}{\hbar^2} |A_{fi}|^2$

$t^2 \sim$ "intensity of resonance"

$\frac{4\pi}{t} \sim$ "width of resonance"

2) Regions of Validity

(A) - Resonant approximation (above),

keep $\frac{\sin \omega_f t/2}{\omega_f - t}$, neglect $\frac{\sin \omega_i t/2}{\omega_i - t}$



$\Delta\omega$ should be much larger than the frequency distance between the two curves

so that no interference occurs:

$$\frac{4\pi}{t} \sim \Delta\omega \ll |\omega_{fi}|$$

$$\Rightarrow t \gg \frac{1}{|\omega_{fi}|} \sim \frac{1}{\omega} \quad \leftarrow \text{resonance validity}$$

(B) - Validity of first-order calculation:

$$P_{i \rightarrow f}(t) \Big|_{\omega = |\omega_{fi}|} = \frac{\lambda^2 t^2}{t^2} |A_{fi}|^2$$

$$B \ll 1 \quad \text{provided} \quad t \ll \frac{\hbar}{\lambda |A_{fi}|} \quad \leftarrow 1^{\text{st}} \text{ order validity}$$

$$(A), (B) \text{ yield that } \frac{\hbar}{\lambda |A_{fi}|} \gg \frac{1}{\omega_{fi}}.$$

Here, we assumed the transitions between discrete eigenstates of $\hat{H}_0$, but what happens if the final state belonged to a continuum spectrum?

Fermi's Golden Rule (Simplified)

Recall the time evolved state $|\tilde{\psi}_i(t)\rangle = |\psi_i\rangle e^{i\frac{E_i(t-t_0)}{\hbar}}$

and define the discrete continuum $|\psi_f\rangle \rightarrow \sum_n |\psi_n\rangle$

$$|\tilde{\psi}_i(t)\rangle = |\psi_i\rangle e^{i\frac{E_i(t-t_0)}{\hbar}} - \frac{i}{\hbar} \sum_n e^{i\frac{E_n(t-t_0)}{\hbar}} \int_{t_0}^t dt' e^{i\omega_{ni}t'} |\psi_n\rangle \langle \psi_n | \hat{H}_1(t) | \psi_i \rangle$$

As before, we consider the periodic perturbation

$$\hat{H}_1(t) = \hat{A} e^{i\omega t} + \hat{A}^\dagger e^{-i\omega t}$$

we take
 $T \rightarrow \infty$ later...

when $\hat{H}_1(t)$ only acts for $[-\frac{T}{2}, \frac{T}{2}]$ where $\frac{T}{2} \gg |t_0|$

Then:

$$|\tilde{\psi}_i(t)\rangle \Big|_{t > \frac{T}{2}} = |\psi_i\rangle e^{i\frac{E_i(t-t_0)}{\hbar}} - \frac{i}{\hbar} \sum_n e^{i\frac{E_n(t-t_0)}{\hbar}} \int_{-T/2}^{T/2} dt' e^{i\omega_{ni}t'}$$

$$|\psi_n\rangle [A_{ni} e^{i\omega t'} + A_{ni}^\dagger e^{-i\omega t'}]$$

$$= |\psi_i\rangle e^{i\frac{E_i(t-t_0)}{\hbar}} - \frac{i}{\hbar} \sum_n e^{i\frac{E_n(t-t_0)}{\hbar}} |\psi_n\rangle$$

$$\times \left[A_{ni} \int_{-T/2}^{T/2} dt' e^{i(\omega_{ni}+\omega)t'} + A_{ni}^\dagger \int_{-T/2}^{T/2} dt' e^{i(\omega_{ni}-\omega)t'} \right]$$

Thus

$$P_{i \rightarrow f}(t > \frac{T}{2}, \text{ if } \psi_i(t_0 < \frac{T}{2}))$$

$$= \frac{\lambda^2}{\hbar^2} \left| A_{fi} \int_{-T/2}^{T/2} dt' e^{i(\omega_{fi} + \omega)t'} + A_{fi}^* \int_{-T/2}^{T/2} dt' e^{i(\omega_{fi} - \omega)t'} \right|^2.$$

Next: $T \rightarrow \infty$, and use $\int_{-\infty}^{\infty} e^{iAt} dt = 2\pi \delta(A)$

$\hookrightarrow \lim_{T \rightarrow \infty} \frac{\sin(\omega T/2)}{\omega/2} = 2\pi \delta(\omega)$

Therefore

$$P_{i \rightarrow f}(t > T, T \rightarrow \infty) = \frac{\lambda^2}{\hbar^2} \left| A_{fi} 2\pi \delta(\omega + \omega_{fi}) + A_{fi}^* 2\pi \delta(\omega - \omega_{fi}) \right|^2$$

which, clearly, only one cannot be zero, so for no loss of generality we choose $\omega = \omega_{fi}$:

$$P_{i \rightarrow f}(T, T \rightarrow \infty) = \frac{\lambda^2}{\hbar^2} |A_{fi}|^2 \cdot 2\pi \delta(\omega - \omega_{fi}) \cdot 2\pi \delta(\omega - \omega_{fi})$$

$2\pi \delta(0) = \int_{-T/2}^{T/2} e^{0 \cdot it'} dt' = T$

or

$$\frac{P_{i \rightarrow f}(T, T \rightarrow \infty)}{T} = \frac{\lambda^2}{\hbar^2} |A_{fi}|^2 2\pi \delta(\omega - \omega_{fi})$$

$$= \frac{\text{transition probability}}{\text{unit time}}$$

Fermi's Golden Rule

How does transition probability / unit time make sense? Some final states are left both perturbed and unperturbed, a smooth density of states, each with specific energy.

Integrating over final states isolates the state $|\psi\rangle$ with energy $\omega = \omega_f$ given by $\delta(\omega - \omega_f)$.

Decay Rate

$$\frac{\text{Transition Probability}}{\text{Unit time}} = \text{rate } (i \rightarrow f) : \Gamma(|\psi_i\rangle \rightarrow |\psi_f\rangle)$$

$$= \frac{\lambda^2}{\hbar^2} 2\pi |A_{if}|^2 \int d\epsilon_f \underbrace{g(\epsilon_f)}_{\substack{\text{\# of states with} \\ \epsilon \in (\epsilon_f, \epsilon_f + d\epsilon_f) \\ \text{with } \epsilon_f = \hbar\omega_f}} \delta(\omega - \omega_f)$$

$$\delta(\omega - \omega_f) \equiv \hbar \delta(\hbar\omega - \hbar\omega_f)$$

$$= \frac{\lambda^2}{\hbar^2} 2\pi |A_{if}|^2 g(\hbar\omega)$$

↑
density of final
states with
 $\epsilon_f = \hbar\omega_f$.

Born Approximation

Idea: $\hat{H}_0 = \frac{\vec{p}^2}{2m}$: free particle

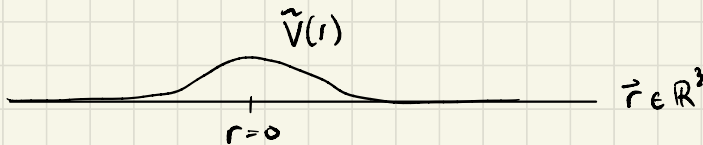
$\hat{H}_1 = \hat{V}(\vec{r})$: scattering potential

and treat $\hat{H}_1$ as perturbation acting from $-\frac{T}{2}$ to $\frac{T}{2}$.

This is indeed time-independent, so instead use

$$\hat{H}_1 = \tilde{V}(r) \frac{e^{i\omega t} + e^{-i\omega t}}{2} \Big|_{\omega \rightarrow 0}$$

In other words,



so $\hat{V}(\vec{r})$ is like a small "bump" at the origin.

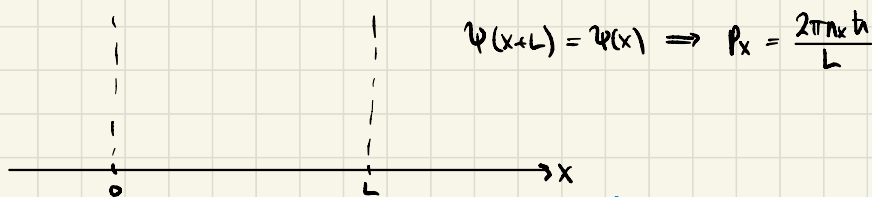
We have

$$P_{i \rightarrow f}(T, T \rightarrow \infty) = \frac{\lambda^2}{\hbar^2} |A_{fi}|^2 2\pi \delta(\omega - \omega_{fi}) \cancel{2\pi \delta(\omega - \omega_{fi})}^T$$

↑
doesn't really make sense for a single state f since $\sim \delta(\omega_{fi})$

and, of course, $\hat{H}_0$ has a continuous spectrum.

How to regulate? Let's try to reduce the problem to a periodic "box" in $\mathbb{R}^1$:



$$\psi(x+L) = \psi(x) \Rightarrow p_x = \frac{2\pi n \hbar}{L}$$

The free particle states are $e^{i \frac{2\pi n}{L} x}$ $\leftarrow \frac{p_x}{\hbar} \in (-\infty, \infty)$

In $\mathbb{R}^3$, we similarly have

$$\begin{aligned} \psi_{\vec{p}} &= C e^{i(\frac{p_x}{\hbar}x + \frac{p_y}{\hbar}y + \frac{p_z}{\hbar}z)} \\ &= C e^{i\vec{p} \cdot \vec{r}/\hbar} \quad \vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z) \end{aligned}$$

Under normalization,

$$1 = \iiint_0^L d^3V c^2 = \overset{\substack{\text{red arrow} \\ \equiv V}}{\int} c^2 \Rightarrow c = \frac{1}{\sqrt{V}}.$$

Our initial state is then

$$|i\rangle \longleftrightarrow \psi_{\vec{p}_i} = \frac{1}{\sqrt{V}} e^{i\vec{p}_i \cdot \vec{r}/\hbar}$$

which forms a continuum in $\vec{p}$ as $L \rightarrow \infty$.

Qn: What is probability that at time $t > T$ the initial state $\psi_{\vec{p}_i}$ transitions into a set of states with

$$\vec{p}_f \in \{\text{small volume in } \vec{p}\text{-space}\}$$

We then can integrate over our final states with density $n(\vec{p}_f)$

$$\int_{\mathbb{R}^3} d^3\vec{p}_f n(\vec{p}_f)$$

where $\int d^3\vec{p} n(\vec{p}) = \text{total \# of states in } V$

In 1-D, $\psi_{p_x} = \frac{1}{\sqrt{L}} e^{i \frac{p_x x}{\hbar}}$, $p_x = \frac{2\pi n_x \hbar}{L}$

changing $\Delta n_x = 1$, $\Delta p_x = \frac{2\pi\hbar}{L}$ so $dp_x = \frac{2\pi\hbar}{L} dn_x$

but we could have another quantity (i.e. a matrix element) which also depends on $p_x = \frac{2\pi\hbar n_x}{L}$ from which we integrate over the various values of n_x :

$$\sum_{n_x=n_{\min}}^{n_{\max}} f(p_x = \frac{2\pi\hbar}{L} n_x) = \sum_{n_x} \Delta n_x f(p_x = \frac{2\pi\hbar}{L} n_x)$$

$$= \sum_{p_{\min}}^{p_{\max}} \frac{L}{2\pi\hbar} \Delta p_x f(p_x)$$

$$\rightarrow \frac{L}{2\pi\hbar} \int_{p_{\min}}^{p_{\max}} dp_x f(p_x)$$

In $\mathbb{R}^3$, easily goes into $\frac{V}{(2\pi\hbar)^3} \int d^3\vec{p} f(\vec{p})$

implying that $\int_{\mathbb{R}^3} d^3\vec{p} \cdot n(\vec{p}_f) d^3\vec{p} = \int_{\mathbb{R}^3} d^3\vec{p} \frac{V}{(2\pi\hbar)^3} d^3\vec{p}$

↑
transition probability
as before...
in any chosen system

↑
over some region in
 $\vec{p}$ -space

$x^* \in \{\text{zeros of } f(x)\}$

$$\delta(f(x)) = \frac{\delta(x-x^*)}{|f'(x^*)|}$$

Thus

$$\int_{\vec{p}_i \rightarrow \vec{p}_f} (T, T \rightarrow \infty) \frac{V}{(2\pi\hbar)^3} d^3\vec{p}_f = T \int \frac{\lambda^2}{T^2} |A_{fi}|^2 2\pi \delta\left(\frac{p_i^2}{2m\hbar} - \frac{p_f^2}{2m\hbar}\right) \frac{V}{(2\pi\hbar)^3} d^3\vec{p}_f$$

$$A_{fi} = \frac{1}{V} \int e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x}/\hbar} \tilde{V}(\vec{x}) d^3\vec{x}$$

$$= \frac{T}{\hbar^2} \frac{2\pi}{(2\pi\hbar)^3} \int d^3\vec{p}_f \frac{V}{V^2} \left| \int d^3\vec{x} e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x}/\hbar} \tilde{V}(\vec{x}) \right|^2 \cdot \delta\left(\frac{\vec{p}_i^2 - \vec{p}_f^2}{2m\hbar}\right)$$

↓
 $p_f^2 dp_f d\Omega_f$
in $\vec{p}_f$ -space

$$= \frac{T}{4\pi^2} \frac{1}{\hbar^4} \frac{m}{V} p_i \int d^2\Omega_f \left| \int d^3\vec{x} e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x}/\hbar} \tilde{V}(\vec{x}) \right|^2$$

integration $\int_0^\infty p_f^2 dp_f \delta(\dots)$
cancels the δ -function
into $|\vec{p}_f| \rightarrow |\vec{p}_i|$.

↓
Solid
cangle

But after $T \rightarrow \infty$, we have

$P(\vec{p}_i \rightarrow \vec{p}_f, \text{ with } |\vec{p}_i| = |\vec{p}_f|, \text{ over } d\Omega_f)$

$$= T \underbrace{\frac{m}{V}}_{\text{time}} \underbrace{\frac{1}{4\pi^2\hbar^4}}_{\text{flux}} \left| \int d^3\vec{x} \tilde{V}(\vec{x}) e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x}/\hbar} \right|^2 d^2\Omega_{p_f}$$

Example: Initial state $\psi_i = \frac{1}{\sqrt{V}} e^{i \vec{p}_i \cdot \vec{x} / \hbar}$

$$\begin{aligned} \text{Probability Current } \vec{j}_i &= \frac{\hbar}{m} |\psi_i|^2 \vec{\nabla} \left(\frac{\vec{p}_i \cdot \vec{x}}{\hbar} \right) \\ &= \frac{\vec{p}_i}{m} |\psi_i|^2 \quad \text{probability current of } i\text{-state.} \end{aligned}$$

$$\text{probability flux} \Rightarrow |\vec{j}_i| = \frac{1}{mV} p_i$$

$$\frac{\text{transition prob. in time } T}{T} = \text{transition prob. in unit time,}$$

$$\text{but } \frac{\text{transition prob. in unit time}}{\text{probability flux}} = \text{diff. cross section } \frac{d\sigma}{d\Omega}$$

Hence

$$P(p_i \rightarrow p_f, |\vec{p}_i| = |\vec{p}_f|, d^2\Omega_{p_f}) = \frac{T \frac{m}{V} p_i}{T \frac{1}{mV} p_i} \frac{1}{4\pi^2 \hbar^4} \left| \int d^3\vec{x} \tilde{V}(\vec{x}) e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x} / \hbar} \right|^2 \times d^2\Omega_{p_f}$$

$$= \frac{m^2}{4\pi^2 \hbar^4} \left| \int d^3\vec{x} \tilde{V}(\vec{x}) e^{i(\vec{p}_i - \vec{p}_f) \cdot \vec{x} / \hbar} \right|^2 d^2\Omega_{p_f}$$

$$= d^2\sigma(\theta, \varphi)$$

↑ Born formula

Validity of Fermi's Golden Rule:

• Perturbation Theory: $T \ll \frac{\hbar}{\lambda |A_{fi}|}$

We also had $T \gg \frac{1}{|W_{fi}|} \sim \frac{1}{\omega}$

Should now be
 $T \gg \frac{\hbar}{\Delta}$,
 Δ : energy resolution

so the validity of resonance approximation is
nonoverlapping.

Probability over # final states around E_f :

$$T \int_{E_f - \Delta}^{E_f + \Delta} dE \left(\frac{\sin \frac{E - E_i}{2\hbar} T}{\frac{E - E_i}{2\hbar} T} \right)^2 \times \left[\begin{array}{l} \text{\# states; slowly varying} \\ \text{function of } E, \sqrt{E} \end{array} \right]$$

Let $x = \frac{E - E_i}{2\hbar} T$ so $dE \cdot T = 2\hbar dx$ so

$$T 2\hbar \int_{\frac{E_f - E_i}{2\hbar} T - \frac{\Delta T}{\hbar}}^{\frac{E_f + E_i}{2\hbar} T + \frac{\Delta T}{\hbar}} dx \left(\frac{\sin x}{x} \right)^2 = 2\pi\hbar T$$

↑ for $T \gg \frac{\hbar}{\Delta}$, $E_f = E_i$

The restrictions $T \gg \frac{\hbar}{\Delta}$, $\ll \frac{\hbar}{\lambda |A_{fi}|}$ are necessary but

necessarily sufficient - need next order term after this.

